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The Hyperbolic Heat Conduction Equation in an Anisotropic Material

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In this work, the phase-lag concept in the wave theory of heat conduction is extended to describe the thermal behavior of an anisotropic material. This is achieved by assuming that there are phase lags of different magnitudes between each component of the heat flux vector and the summation of temperature gradients in all directions of the orthogonal coordinate system. Also, expressions are provided to specify the locations of the principal coordinate axes, the principal thermal conductivities, and the principal thermal relaxation times.

KEY WORDS: anisotropic materials; heat conduction; hyperbolic conduction model; wave conduction model.

1. INTRODUCTION

Hyperbolic heat transport has been receiving increasing attention both for theoretical motivations (analysis of thermal waves and second sound in dielectric solids, finite speed of heat transport, etc.) and for the analysis of some practical problems involving a fast supply of thermal energy (for instance, by a laser pulse or a chemical explosion, etc.). The usual theory of thermal conduction, based on the Fourier law, implies an immediate response to a temperature gradient and leads to a parabolic differential equation for the evolution of the temperature. In contrast, when relaxational effects are taken into account in the constitutive equation describing the heat flux, as, for instance, in the Maxwell–Cattaneo equation, one has a hyperbolic equation which implies a finite speed for heat transport. The literature in this field is rather vast. We refer the reader to several reviews and papers on this subject [1–3].

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To account for the phenomena involving the finite propagation velocity of the thermal wave, the classical Fourier heat flux should be modified. Cattaneo [4] and Vermotte [5] suggested independently a modified heat flux model in the form of

$$q(x, t+\tau) = -k\,\frac{\partial T(x, t)}{\partial x} \tag{1}$$

where q is the heat flux vector, k is the thermal conductivity, and τ is the relaxation time. The constitutive Eq. (1) assumes that the heat flux vector (the effect) and the temperature gradient (the cause) across a material volume occur at different instants of time, and the time delay between the heat flux and the temperature gradient is the relaxation time τ. The first-order expansion of q in Eq. (1) with respect to t connects all the physical quantities at the same time. It results in the expansion

$$\tau\,\frac{\partial q(x, t)}{\partial t} + q(x, t) = -k\,\frac{\partial T(x, t)}{\partial x} \tag{2}$$

In Eq. (2) it is assumed that τ is small enough so that the first-order Taylor expansion of $q(x, t+\tau)$ is an accurate representation for the convection heat flux. The equation of energy conservation for such problems is given as

$$\rho c\,\frac{\partial T}{\partial t} = -\frac{\partial q}{\partial x} \tag{3}$$

where ρ is the density and c is the specific heat. Elimination of q between Eq. (2) and Eq. (3) leads to the hyperbolic heat conduction equation

$$\rho c\tau\,\frac{\partial^2 T}{\partial t^2} + \rho c\,\frac{\partial T}{\partial t} = k\,\frac{\partial^2 T}{\partial x^2} \tag{4}$$

Various analytical and numerical methods (Vick and Ozisik [6]; Yuen and Lee [7]; Kim et al. [8]; Chen and Lin [9]; Ozisik and Tzou [3]; Kozlowska et al. [10]; Tang and Araki [11]) have been proposed to solve the hyperbolic heat conduction problems under different applications and in different configurations.

However, the hyperbolic heat conduction model, in the present form, cannot be used to describe the thermal behavior of anisotropic solids in which the thermal conductivity varies with direction. Heat conduction in anisotropic materials has numerous important applications in various branches of science and engineering. For example, crystals, wood, sedimentary rocks, metals that have undergone heavy cold pressing, laminated sheets, cables, heat shielding materials for space vehicles, fiber-reinforced composite structures, and many others are anisotropic materials.

The aim of the present work is to extend the phase-lag concept in the wave theory of heat conduction to describe the thermal behavior of anisotropic materials. It is assumed that a phase lag exists between each of the conduction fluxes, in a given direction, as for the temperature gradients in all spatial directions.

2. ANALYSIS

The energy equation that describes the thermal behavior of anisotropic, and isotropic, materials is given as

$$-\frac{\partial q_1}{\partial x_1} - \frac{\partial q_2}{\partial x_2} - \frac{\partial q_3}{\partial x_3} + g = \rho c \frac{\partial T}{\partial t} \tag{5}$$

The components of the heat flux vector, in the case of anisotropic solids, depend, in general, on a linear combination of the temperature gradients along the three perpendicular directions:

$$\begin{aligned} -q_1(t+\tau_1) &= k_{11}\frac{\partial T(t)}{\partial x_1} + k_{12}\frac{\partial T(t)}{\partial x_2} + k_{13}\frac{\partial T(t)}{\partial x_3} \\ -q_2(t+\tau_2) &= k_{21}\frac{\partial T(t)}{\partial x_1} + k_{22}\frac{\partial T(t)}{\partial x_2} + k_{23}\frac{\partial T(t)}{\partial x_3} \\ -q_3(t+\tau_3) &= k_{31}\frac{\partial T(t)}{\partial x_1} + k_{32}\frac{\partial T(t)}{\partial x_2} + k_{33}\frac{\partial T(t)}{\partial x_3} \end{aligned} \tag{6}$$

where in Eqs. (6) it is assumed that the heat flux vector (the effect) and the temperature gradient (the cause) across a material volume occur at different instants of time and the time delays between each heat flux in a given direction and the combinations of temperature gradients in all directions are the relaxation times τ_1, τ_2, and τ_3. It is reasonable here to assume that different heat fluxes have different thermal relaxation times since the material is anisotropic. The first-order expansions of q_1, q_2, and q_3 in Eqs. (6) with respect to t connects all the physical quantities at the same time. It results in the expansion

$$-q_1(t)-\tau_1\frac{\partial q_1(t)}{\partial t}=k_{11}\frac{\partial T(t)}{\partial x_1}+k_{12}\frac{\partial T(t)}{\partial x_2}+k_{13}\frac{\partial T(t)}{\partial x_3}$$
$$-q_2(t)-\tau_2\frac{\partial q_2(t)}{\partial t}=k_{21}\frac{\partial T(t)}{\partial x_1}+k_{22}\frac{\partial T(t)}{\partial x_2}+k_{23}\frac{\partial T(t)}{\partial x_3} \quad (7)$$
$$-q_3(t)-\tau_3\frac{\partial q_3(t)}{\partial t}=k_{31}\frac{\partial T(t)}{\partial x_1}+k_{32}\frac{\partial T(t)}{\partial x_2}+k_{33}\frac{\partial T(t)}{\partial x_3}$$

Now Eqs. (5) and (7) describe the thermal behavior of an anisotropic material when the speed of heat transport is finite.

Usually, it is better to express the energy equation, Eq. (5), in terms of temperature only and this may be achieved by eliminating the components of the heat flux vector q between Eq. (5) and Eq. (7). Now derive Eqs. (7) with respect to x_1, x_2, and x_3, respectively, and combine the results to yield

$$-\frac{\partial q_1}{\partial x_1}-\frac{\partial q_2}{\partial x_2}-\frac{\partial q_3}{\partial x_3}-\frac{\partial}{\partial t}\left[\tau_1\frac{\partial q_1}{\partial x_1}+\tau_2\frac{\partial q_2}{\partial x_2}+\tau_3\frac{\partial q_3}{\partial x_3}\right]$$
$$=k_{11}\frac{\partial^2 T(t)}{\partial x_1^2}+k_{12}\frac{\partial^2 T(t)}{\partial x_1\,\partial x_2}+k_{13}\frac{\partial^2 T(t)}{\partial x_1\,\partial x_3}$$
$$+k_{21}\frac{\partial^2 T(t)}{\partial x_1\,\partial x_2}+k_{22}\frac{\partial^2 T(t)}{\partial x_2^2}+k_{23}\frac{\partial^2 T(t)}{\partial x_2\,\partial x_3}$$
$$+k_{31}\frac{\partial^2 T(t)}{\partial x_1\,\partial x_3}+k_{32}\frac{\partial^2 T(t)}{\partial x_2\,\partial x_3}+k_{33}\frac{\partial^2 T(t)}{\partial x_3^2} \quad (8)$$

The only possible way to eliminate the gradient of the heat flux vector from Eq. (8) is to assume that $\tau_1=\tau_2=\tau_3=\tau$ and then substitute for the gradient of the heat vector from Eq. (5), to yield

$$\rho c\frac{\partial T}{\partial t}+\rho c\tau\frac{\partial^2 T}{\partial t^2}-g-\tau\frac{\partial g}{\partial t}$$
$$=k_{11}\frac{\partial^2 T(t)}{\partial x_1^2}+k_{12}\frac{\partial^2 T(t)}{\partial x_1\,\partial x_2}+k_{13}\frac{\partial^2 T(t)}{\partial x_1\partial x_3}$$
$$+k_{21}\frac{\partial^2 T(t)}{\partial x_1\,\partial x_2}+k_{22}\frac{\partial^2 T(t)}{\partial x_2^2}+k_{23}\frac{\partial^2 T(t)}{\partial x_2\partial x_3}$$
$$+k_{31}\frac{\partial^2 T(t)}{\partial x_1\,\partial x_3}+k_{32}\frac{\partial^2 T(t)}{\partial x_2\,\partial x_3}+k_{33}\frac{\partial^2 T(t)}{\partial x_3^2} \quad (9)$$

In situations in which the assumption that $\tau_1 = \tau_2 = \tau_3$ is not valid, another approach will be followed. In this approach, the phase lag between the components of the heat flux vector and the temperature gradients in different directions, is expressed as

$$\begin{aligned}
-q_1(t) &= k_{11}\frac{\partial T(t+\tau_{11})}{\partial x_1} + k_{12}\frac{\partial T(t+\tau_{12})}{\partial x_2} + k_{13}\frac{\partial T(t+\tau_{13})}{\partial x_3} \\
-q_2(t) &= k_{21}\frac{\partial T(t+\tau_{21})}{\partial x_1} + k_{22}\frac{\partial T(t+\tau_{22})}{\partial x_2} + k_{23}\frac{\partial T(t+\tau_{23})}{\partial x_3} \\
-q_3(t) &= k_{31}\frac{\partial T(t+\tau_{31})}{\partial x_1} + k_{32}\frac{\partial T(t+\tau_{32})}{\partial x_2} + k_{33}\frac{\partial T(t+\tau_{33})}{\partial x_3}
\end{aligned} \tag{10}$$

Equations (10) are special cases of the dual-phase-lag model, which allows either the temperature gradient (cause) to precede the heat flux vector (effect) or the heat flux vector (cause) to precede the temperature gradient (effect) in the transient process. In Eqs. (10), if the τ's are negative, this implies that the temperature gradients (cause) precede the heat fluxes (effect), and if the τ's are positive, this implies that the heat flux vector (cause) precedes the temperature gradient (effect). The first-order expansion of temperature gradients with respect to τ's in Eq. (10) connects all the physical quantities at the same instant of time. It results in the expressions

$$\begin{aligned}
-q_1 &= k_{11}\frac{\partial T}{\partial x_1} + k_{11}\tau_{11}\frac{\partial^2 T}{\partial x_1\,\partial t} + k_{12}\frac{\partial T}{\partial x_2} \\
&\quad + k_{12}\tau_{12}\frac{\partial^2 T}{\partial x_2\,\partial t} + k_{13}\frac{\partial T}{\partial x_3} + k_{13}\tau_{13}\frac{\partial^2 T}{\partial x_3\,\partial t} \\
-q_2 &= k_{21}\frac{\partial T}{\partial x_1} + k_{21}\tau_{21}\frac{\partial^2 T}{\partial x_1\,\partial t} + k_{22}\frac{\partial T}{\partial x_2} \\
&\quad + k_{22}\tau_{22}\frac{\partial^2 T}{\partial x_2\,\partial t} + k_{23}\frac{\partial T}{\partial x_3} + k_{23}\tau_{23}\frac{\partial^2 T}{\partial x_3\,\partial t} \\
-q_3 &= k_{31}\frac{\partial T}{\partial x_1} + k_{31}\tau_{31}\frac{\partial^2 T}{\partial x_1\,\partial t} + k_{32}\frac{\partial T}{\partial x_2} \\
&\quad + k_{32}\tau_{32}\frac{\partial^2 T}{\partial x_2\,\partial t} + k_{33}\frac{\partial T}{\partial x_3} + k_{33}\tau_{33}\frac{\partial^2 T}{\partial x_3\,\partial t}
\end{aligned} \tag{11}$$

Now derive Eqs. (11) with respect to x_1, x_2, and x_3, respectively, and substitute for $\partial q_1/\partial x_1$, $\partial q_2/\partial x_2$, and $\partial q_3/\partial x_3$ into the energy equation to yield

$$\begin{aligned}\rho c\frac{\partial T}{\partial t}&=k_{11}\frac{\partial^2 T}{\partial x_1^2}+2k_{12}\frac{\partial^2 T}{\partial x_1\,\partial x_2}+2k_{13}\frac{\partial^2 T}{\partial x_1\,\partial x_3}\\&\quad+k_{22}\frac{\partial^2 T}{\partial x_2^2}+2k_{23}\frac{\partial^2 T}{\partial x_2\,\partial x_3}+k_{33}\frac{\partial^2 T}{\partial x_3^2}\\&\quad+k_{11}\tau_{11}\frac{\partial^3 T}{\partial t\,\partial x_1^2}+2k_{12}\tau_{12}\frac{\partial^3 T}{\partial t\,\partial x_1\,\partial x_2}+2k_{13}\tau_{13}\frac{\partial^3 T}{\partial t\,\partial x_1\,\partial x_3}\\&\quad+k_{22}\tau_{22}\frac{\partial^3 T}{\partial t\,\partial x_2^2}+2k_{23}\tau_{23}\frac{\partial^3 T}{\partial t\,\partial x_2\,\partial x_3}+k_{33}\tau_{33}\frac{\partial^3 T}{\partial t\,\partial x_3^2}+g\end{aligned}\tag{12}$$

Equation (12) may be rewritten in terms of the principal coordinate axes ζ_1, ζ_2, and ζ_3 as

$$\left(1+\frac{\partial}{\partial t}\right)\frac{\partial^2 T}{\partial \zeta_1^2}+\left(1+\frac{\partial}{\partial t}\right)\frac{\partial^2 T}{\partial \zeta_2^2}+\left(1+\frac{\partial}{\partial t}\right)\frac{\partial^2 T}{\partial \zeta_3^2}+g=\rho c\frac{\partial T}{\partial t}\tag{13}$$

The principal axes ζ_1, ζ_2, and ζ_3 are determined in the following manner: if l_1, l_2, and l_3 are the directional cosines of the principal axis $o\zeta_1$ with respect to the axes ox_1, ox_2, and ox_3, and $\lambda_1=k_1(1+\tau_1)$ is the principal value along the direction $o\zeta_1$, then l_1, l_2, and l_3 satisfy the relation

$$\begin{pmatrix}k_{11}(1+\tau_{11})-\lambda_1 & k_{12}(1+\tau_{12}) & k_{13}(1+\tau_{13})\\ k_{21}(1+\tau_{21}) & k_{22}(1+\tau_{22})-\lambda_1 & k_{23}(1+\tau_{23})\\ k_{31}(1+\tau_{31}) & k_{32}(1+\tau_{32}) & k_{33}(1+\tau_{33})-\lambda_1\end{pmatrix}\begin{bmatrix}l_1\\ l_2\\ l_3\end{bmatrix}=0\tag{14}$$

which provides three homogeneous equations for the three unknowns l_1, l_2, and l_3. Only two of these equations are linearly independent. An additional relation is obtained from the requirement that the directional cosines satisfy

$$l_1^2+l_2^2+l_3^2=1$$

Thus, the three directional cosines of the principal axis $o\zeta_1$ are determined from the determinantal equation resulting from the matrix Eq. (14). The procedure is repeated with $\lambda_2=k_2(1+\tau_2)$ for the determination of the principal axis $o\zeta_2$ and with $\lambda_3=k_3(1+\tau_3)$ for the principal axis $o\zeta_3$.

The principal values $\lambda_1=k_1(1+\tau_1)$, $\lambda_2=k_2(1+\tau_2)$, and $\lambda_3=k_3(1+\tau_3)$ are determined as three roots of the following equation:

$$\begin{vmatrix}k_{11}(1+\tau_{11})-\lambda & k_{12}(1+\tau_{12}) & k_{13}(1+\tau_{13})\\ k_{21}(1+\tau_{21}) & k_{22}(1+\tau_{22})-\lambda & k_{23}(1+\tau_{23})\\ k_{31}(1+\tau_{31}) & k_{32}(1+\tau_{32}) & k_{33}(1+\tau_{33})-\lambda\end{vmatrix}=0\tag{15}$$

3. CONCLUSION

The hyperbolic heat conduction model was extended to describe the thermal behavior of an anisotropic material. By assuming different phase lags between each component of the heat flux vector and the summation of temperature gradients in all directions of the orthogonal coordinate system, the relaxational effects in different directions were taken into account.

NOMENCLATURE

c	Specific heat capacity
g	Heating source
k	Thermal conductivity
k_{ij}	Conductivity coefficient
k_i	Principal conductivity
l_i	Direction cosine of the principal axis $o\zeta_i$
q_i	Component of the conduction heat flux vector
t	Time
T	Temperature
x_i	Spatial coordinate

Greek Letters

λ_i	Principal combination, $k_i(1+\tau_i)$
ρ	Density
τ	Thermal relaxation time
τ_{ij}	Components of thermal relaxation time
ζ_i	Axes of the principal coordinate system

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